tion will differ only slightly from the zero-order wave function, and the true eigenvalue will differ only slightly from the zero-order eigenvalue.

Figure 7.3b shows the correct shape for the true eigenfunction. The shape can be derived qualitatively by simple arguments. Near $x = L/2$, and without

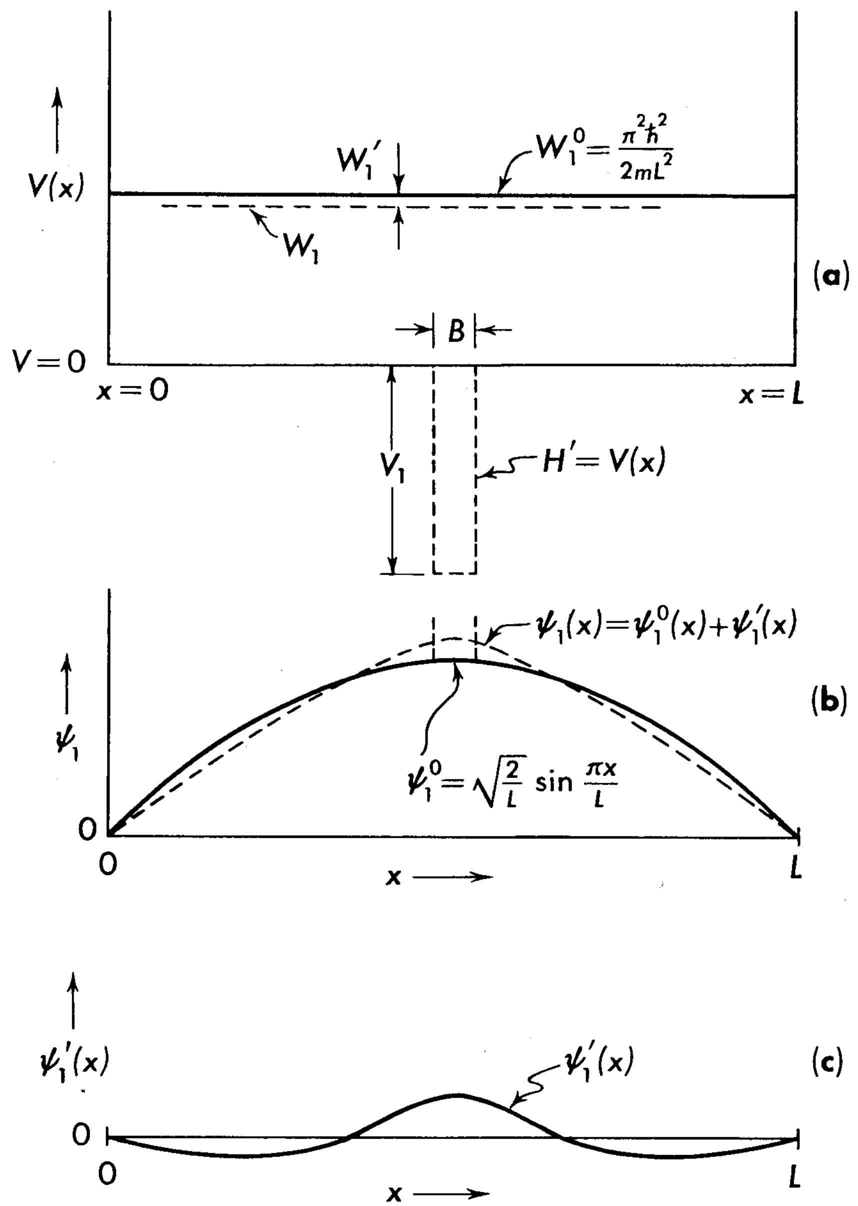


Fig. 7.3. A one-dimensional system containing a small, central potential well.

the perturbing well, the curvature of ψ, $(d^2\psi/dx^2)$ is nearly constant. When the new well is added, the curvature of ψ in the region B must be considerably greater than it was before, and therefore greater than the curvature just outside the well. This occurs since, in the region B, the difference between the potential energy and the total energy is much greater. Inside the region B the true wave

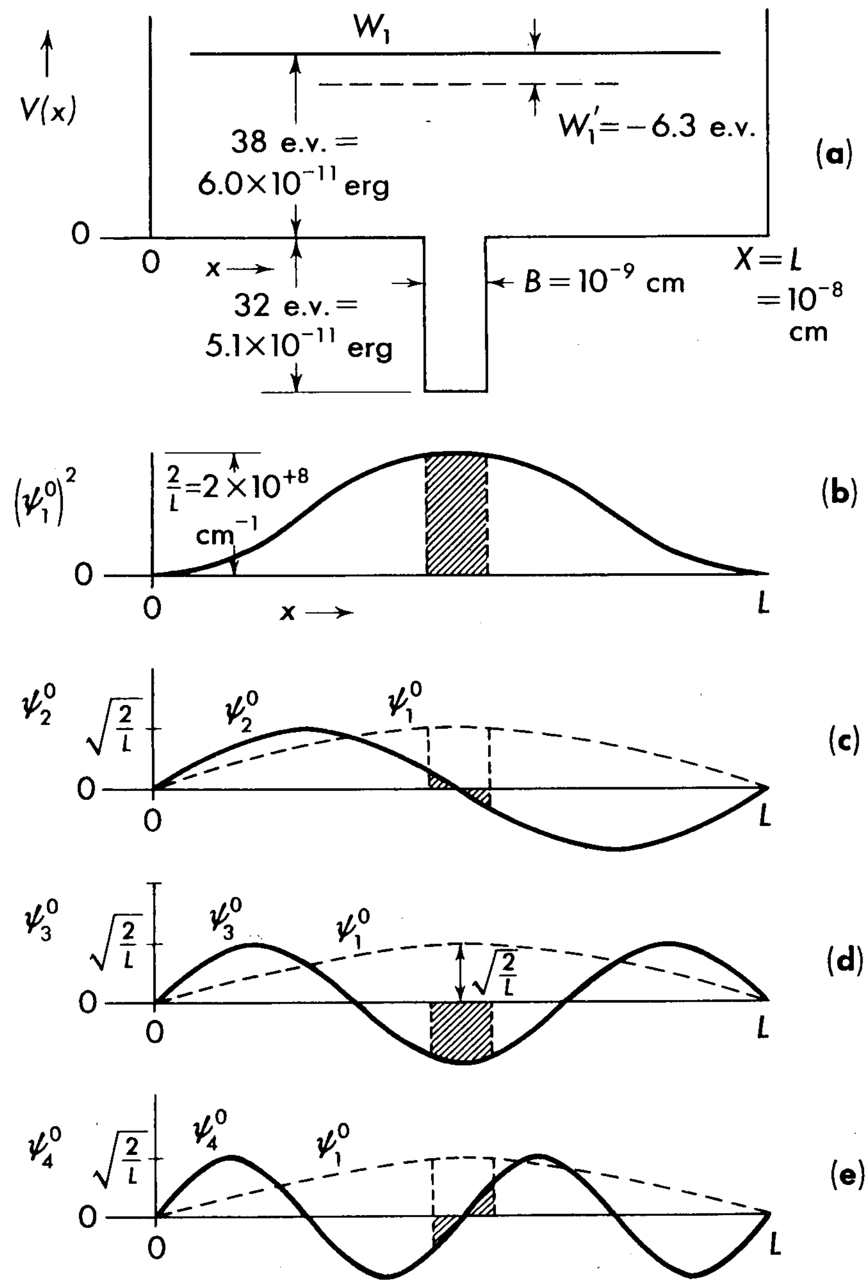


Fig. 7.4. A sample calculation using perturbation theory.

and the energy eigenvalues are,

$$W_n^0 = n^2\pi^2\hbar^2/2mL^2$$

Let the mass $= 9.11 \times 10^{-28}$ gm, $L = 10^{-8}$ cm. Since $\hbar = 1.054 \times 10^{-27}$ erg sec, we have

$$\psi_n^0 = \sqrt{2 \times 10^8} \sin n\pi x/10^{-8}\ (\text{cm})^{-(1/2)}$$

The lowest energy level is[5]

$$W_1^0 = 6{\cdot}0 \times 10^{-11} \text{ erg, or } 38 \text{ e.v.}$$

[5] If $\hbar = 1.054 \times 10^{-34}$ joule sec, $m = 9.11 \times 10^{-31}$ kg, and $L = 10^{-10}$ m, then $W_1^0 = 6.0 \times 10^{-18}$ joule (1 e.v. $= 1.6 \times 10^{-19}$ joule).

With the aid of Figure 7.4d, one can see at once that $H'_{41} = 0$, and therefore $a_4=0$.

As higher a_j's are calculated, one should use exact integration in the calculation of the intensity of the odd-numbered components, because the eigenfunctions vary more rapidly inside the perturbing well, although by symmetery

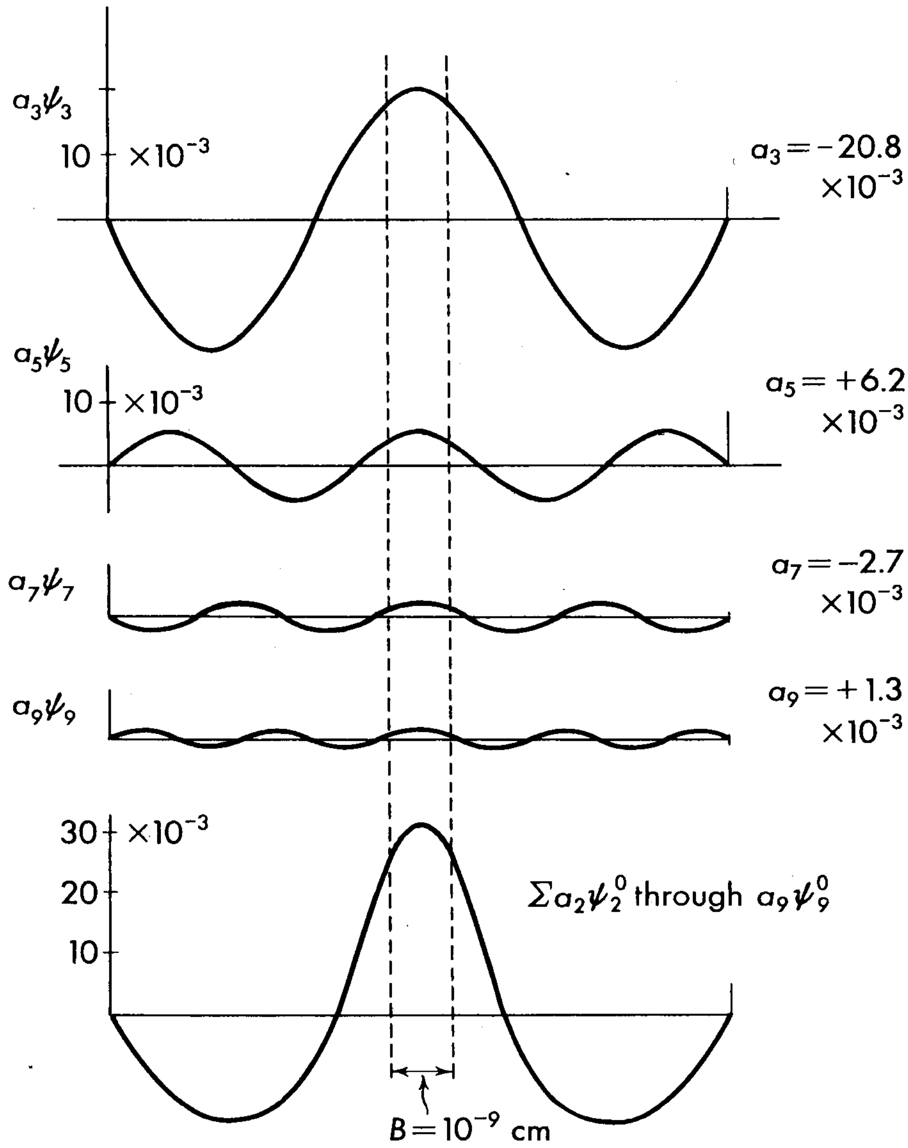
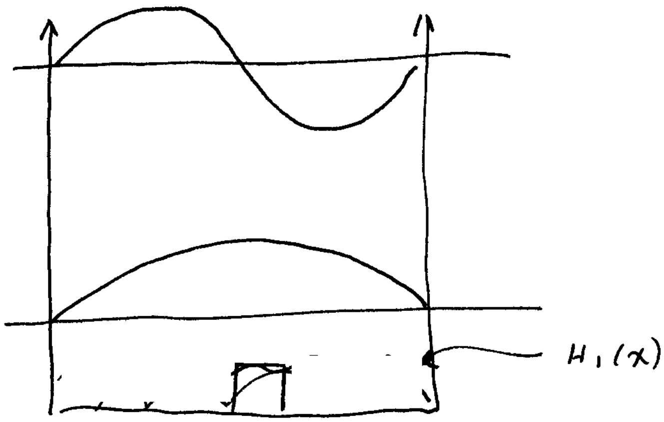


Fig. 7.5. The calculated corrections to the zero-order state ψ_1^0 of the system of Figure 7·4.

all of the even-numbered components are always exactly zero. Because the denominator $W_j^0 - W_1^0$ appears in the calculation of a_j, the magnitude of a_j becomes smaller with increasing $W_j^0 - W_1^0$.

Continuing the calculation of the a_j's, we find the amplitude of the terms up through $n = 9$. These are shown in Figure 7.5. The component wave functions are drawn to scale, with the correct sign. At the bottom of Figure

EXAMPLE 1: THE PERTURBED SQUARE WELL



$$\langle x | n \rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = n^2 \left(\frac{h^2}{8ma^2}\right)$$

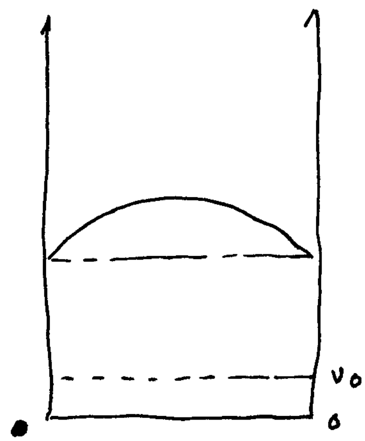
$$E_n^{(1)} = E_n + \langle n | H_1 | n \rangle$$

$$E_n^{(1)} = E_n + \int_0^a H_1(x) \frac{2}{a} \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$E_1^{(1)} = E_1 + \Delta E_1$$

$$E_2^{(1)} = E_2 + \Delta E_2$$

THE WHOLE ~~POTENTIAL~~ BRICK



SHIFT IN GROUND STATE ENERGY

$$\Delta E_1 = \int_0^a H_1(x)\, \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right) dx$$

($H_1(x) \uparrow V_0$)

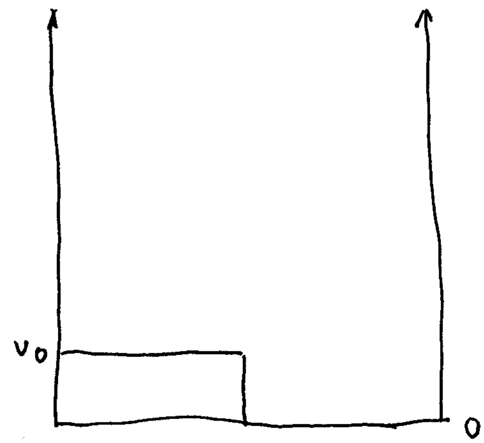
$$\Delta E_1 = V_0$$

FOR ANY STATE n

$$\Delta E_n = V_0$$

"FO TIPT GIVES THE EXACT ANSWER"

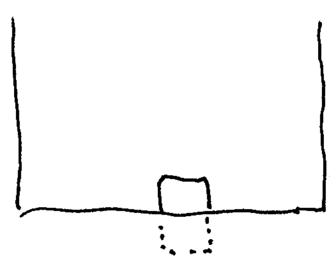
THE HALF BRICK



$$\Delta E_m = \int_0^{a/2} V_0 \, \frac{2}{a} \sin^2\left(\frac{m\pi x}{a}\right) dx$$

$$= \frac{V_0}{2}$$

THE TENTH BRICK



$$\Delta E_m = \int_{-a/20}^{a/20} V_0 \, \frac{2}{a} \sin^2\left(\frac{m\pi x}{a}\right) dx$$

APPLY TIPT TO HYDROGEN

ELECTRONIC STRUCTURE $-\frac{13.6 \text{ eV}}{n^2}$

FINE STRUCTURE $\vec{L} \cdot \vec{S}$ AND RELATIVITY

HYPERFINE STRUCTURE $\vec{L}$ $\vec{S}$ $\vec{I}$

MATRIX ELEMENTS

$$\langle n | H_1 | n \rangle = \Delta E$$ energy shifts

ENERGY CORRECTIONS

TABLE 6.1: Hierarchy of corrections to the Bohr energies of hydrogen.

Bohr energies:	of order	$\alpha^2 mc^2$
Fine structure:	of order	$\alpha^4 mc^2$
Lamb shift:	of order	$\alpha^5 mc^2$
Hyperfine splitting:	of order	$(m/m_p)\alpha^4 mc^2$

nucleus: Just replace m by the reduced mass (Problem 5.1). More significant is the so-called **fine structure**, which is actually due to two distinct mechanisms: a **relativistic correction**, and **spin-orbit coupling**. Compared to the Bohr energies (Equation 4.70), fine structure is a tiny perturbation—smaller by a factor of α^2, where

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \cong \frac{1}{137.036} \qquad [6.43]$$

is the famous **fine structure constant**. Smaller still (by another factor of α) is the **Lamb shift**, associated with the quantization of the electric field, and smaller by yet another order of magnitude is the **hyperfine structure**, which is due to the magnetic interaction between the dipole moments of the electron and the proton. This hierarchy is summarized in Table 6.1. In the present section we will analyze the fine structure of hydrogen, as an application of time-independent perturbation theory.

Problem 6.11

(a) Express the Bohr energies in terms of the fine structure constant and the rest energy (mc^2) of the electron.

(b) Calculate the fine structure constant from first principles (i.e., without recourse to the empirical values of ϵ_0, e, $\hbar$, and c). *Comment:* The fine structure constant is undoubtedly the most fundamental pure (dimensionless) number in all of physics. It relates the basic constants of electromagnetism (the charge of the electron), relativity (the speed of light), and quantum mechanics (Planck's constant). If you can solve part (b), you have the most certain Nobel Prize in history waiting for you. But I wouldn't recommend spending a lot of time on it right now; many smart people have tried, and all (so far) have failed.

6.3.1 The Relativistic Correction

The first term in the Hamiltonian is supposed to represent kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m}, \qquad [6.44]$$

11.7 THE ENERGY LEVELS OF HYDROGEN, INCLUDING FINE STRUCTURE, THE LAMB SHIFT, AND HYPERFINE SPLITTING

Adding the energy shifts (11.89), (11.111), and (11.114) together, we obtain

$$E_K^{(1)} + E_{\text{S-O}}^{(1)} + E_{\text{D}}^{(1)} = E_{n,j}^{(1)} = -\frac{m_e c^2 (Z\alpha)^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \qquad (11.118$$

Notice that the magnitude of the total energy shift is of order $(Z\alpha)^2$ times th unperturbed energy (10.34) of the atom. In particular, for hydrogen ($Z = 1$), th energy shift is roughly 10^{-5} as large as the unperturbed energy. Thus the per turbations do indeed contribute a *fine structure* to the energy levels—hence th potential is solved exactly.[11]

In 1947 W. E. Lamb and R. C. Retherford observed a very small energy difference between the $2s_{1/2}$ and the $2p_{1/2}$ levels through the absorption of microwave radiation with a frequency of 1058 MHz, corresponding to an energy splitting of 4.4×10^{-6} eV (see Fig. 11.10*b*).[12] This *Lamb shift*, which is of the order $m_e c^2 (Z\alpha)^4 \alpha \log \alpha$, can be explained by quantum electrodynamics in terms of the interaction of the electron with the quantized electromagnetic field.[13] The

---

[11] The exact energy eigenvalues for the Dirac equation with a Coulomb potential are given by

$$E_{n,j} = m_e c^2 \left\{ \left[1 + \left(\frac{Z\alpha}{n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2}} \right)^2 \right]^{-1/2} - 1 \right\}$$

[12] W. E. Lamb and R. C. Retherford, *Phys. Rev.* 72, 241 (1947); **86**, 1014 (1952). This latter paper contains their most precise results. Lamb received the Nobel prize in 1953 for this work.

[13] We examine the quantized electromagnetic field in Chapter 14. However, we will not attempt to work out the value of the Lamb shift, which is itself a taxing problem. For an interesting discussion of the difficulties that this calculation presented to R. P. Feynman and H. Bethe, two of the more clever physicists at performing calculations, see Feynman's Nobel prize speech in *Nobel Lectures—Physics*, vol. III, Elsevier Publications, New York, 1972.

Lamb shift has been measured to five significant figures, providing one of the most sensitive tests of quantum electrodynamics (QED). Note that the magnitude of the Lamb shift is roughly 10^{-6} of the spacing between levels that produce the Balmer series. Thus, measuring the shift itself with an accuracy of one part in 10^5 by detecting the difference in wavelength of visible photons emitted as the atom makes transitions from higher energy states to the $2s_{1/2}$ or $2p_{1/2}$ states would require a resolution of 1 part in 10^{11}! The main reason that we can isolate these QED corrections experimentally with such precision is the fortunate degeneracy,

The fine structure of hydrogen (DARWIN)

According to special relativity, the kinetic energy (*i.e.*, the difference between the total energy and the rest mass energy) of a particle of rest mass m and momentum p is

$$T = \sqrt{p^2\,c^2 + m^2\,c^4} - m\,c^2. \tag{945}$$

In the non-relativistic limit $p \ll m\,c$, we can expand the square-root in the above expression to give

$$T = \frac{p^2}{2\,m}\left[1 - \frac{1}{4}\left(\frac{p}{m\,c}\right)^2 + {\cal O}\left(\frac{p}{m\,c}\right)^4\right]. \tag{946}$$

Hence,

$$T \simeq \frac{p^2}{2\,m} - \frac{p^4}{8\,m^3\,c^2}. \tag{947}$$

Of course, we recognize the first term on the right-hand side of this equation as the standard non-relativistic expression for the kinetic energy. The second term is the lowest-order relativistic correction to this energy. Let us consider the effect of this type of correction on the energy levels of a hydrogen atom. So, the unperturbed Hamiltonian is given by Eq. (890), and the perturbing Hamiltonian takes the form

$$H_1 = -\frac{p^4}{8\,m^3\,c^2}. \tag{948}$$

Now, according to standard first-order perturbation theory (see Sect. 12.4), the lowest-order relativistic correction to the energy of a hydrogen atom state characterized by the standard quantum numbers n, l, and m is given by where $\mathbf{L} = m_e\,\mathbf{r}\times\mathbf{v}$ is the electron's orbital angular momentum. This effect is known as *spin-orbit coupling*. It turns out that the above expression is too large, by a factor 2, due to an obscure relativistic effect known as *Thomas precession*. Hence, the true spin-orbit correction to the Hamiltonian is

$$H_1 = \frac{e^2}{8\pi\,\epsilon_0\,m_e^2\,c^2\,r^3}\,\mathbf{L}\cdot\mathbf{S}. \tag{959}$$

Let us now apply perturbation theory to the hydrogen atom, using the above expression as the perturbing Hamiltonian.

Now

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{960}$$

is the total angular momentum of the system. Hence,

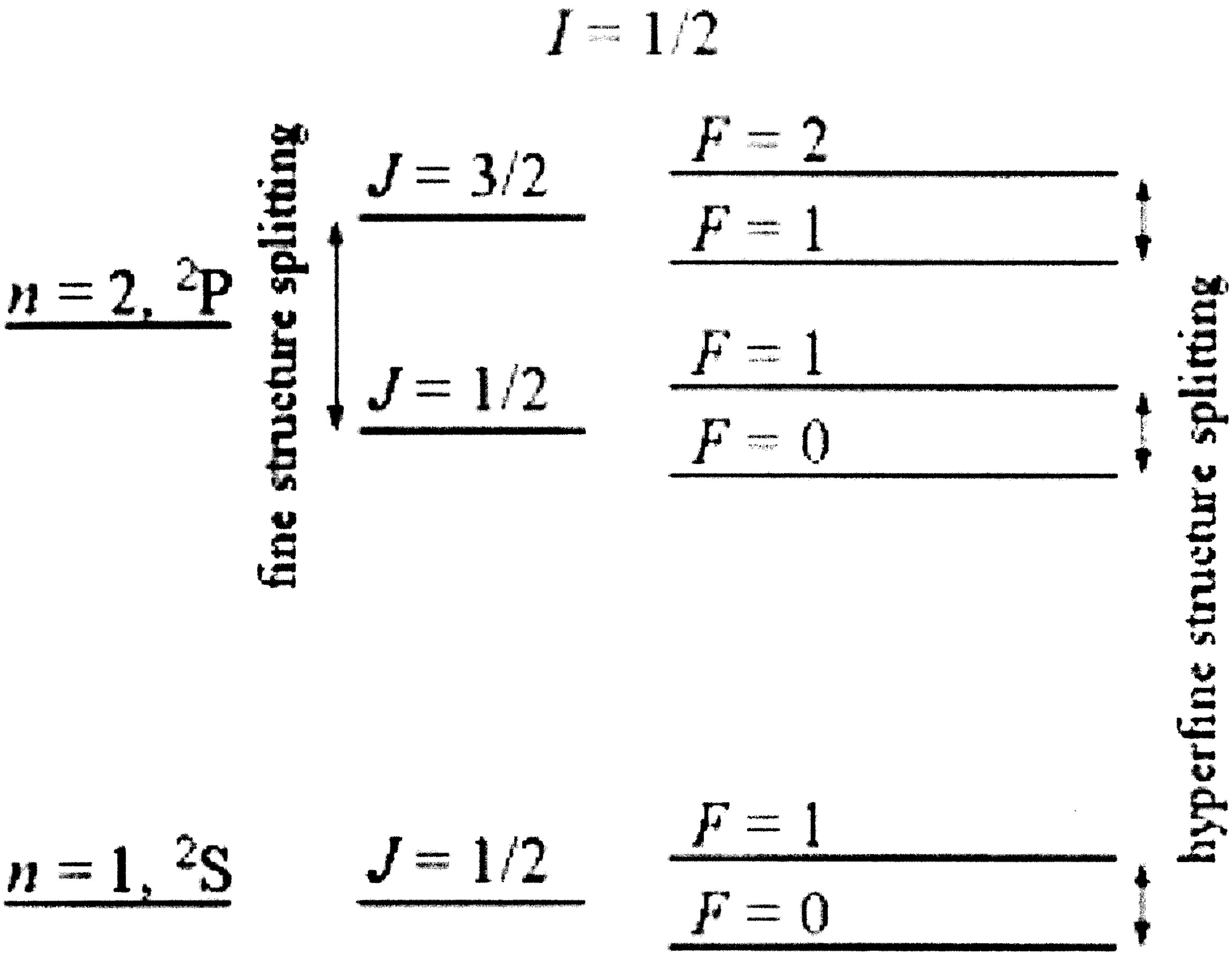
$$J^2 = L^2 + S^2 + 2\,\mathbf{L}\cdot\mathbf{S}, \tag{961}$$

giving

$$\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}\,(J^2 - L^2 - S^2). \tag{962}$$

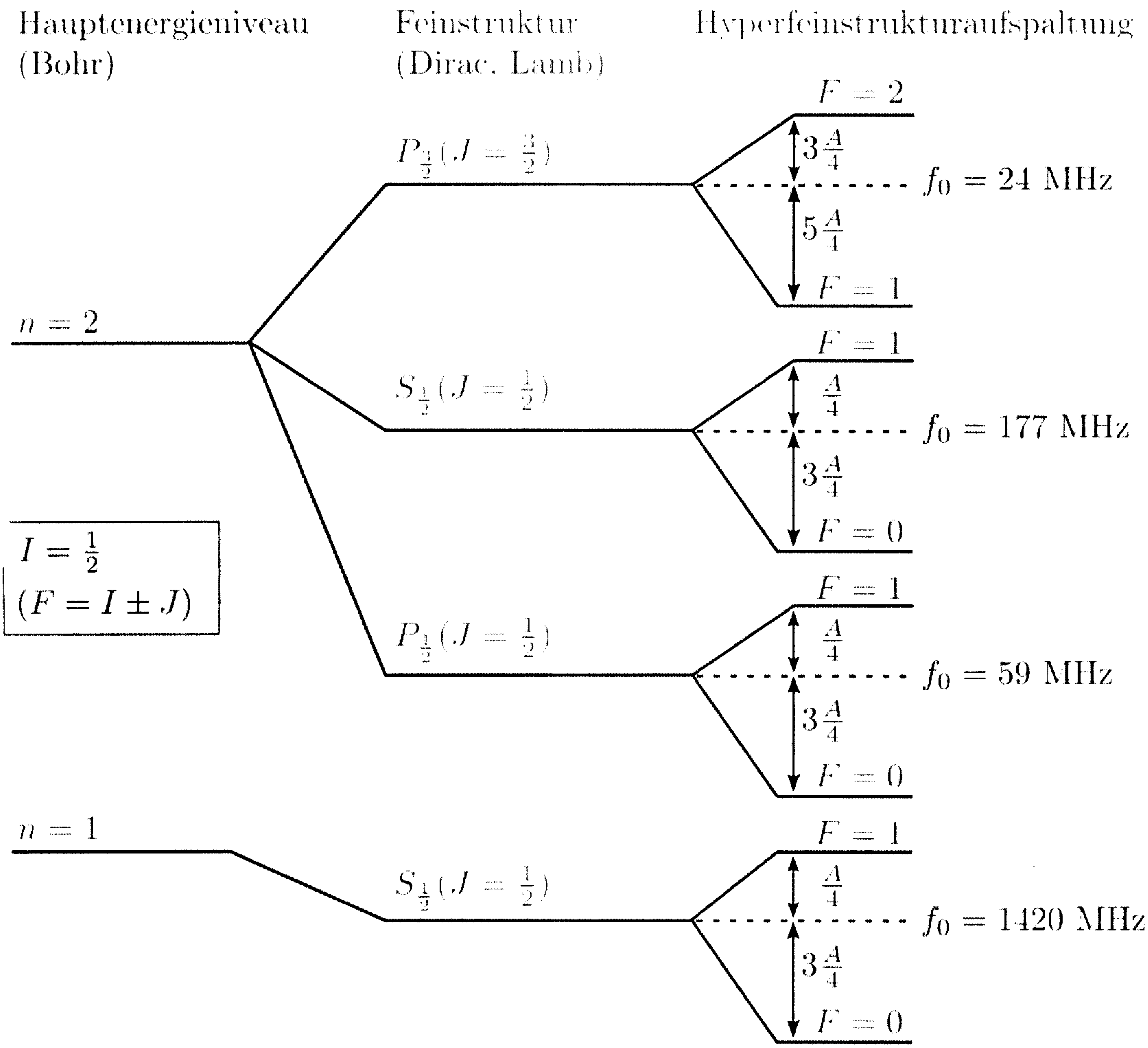
Recall, from Sect. 11.2, that whilst J^2 commutes with both L^2 and S^2, it does not commute with either L_z or S_z. It follows that the perturbing Hamiltonian (959) also commutes with both L^2 and S^2, but does not commute with either L_z or S_z. Hence, the simultaneous eigenstates of the unperturbed Hamiltonian (890) and the perturbing Hamiltonian (959) are the same as the simultaneous eigenstates of L^2, S^2, and J^2 discussed in Sect. 11.3. It is important to know this since, according to Sect. 12.6, we can only safely apply perturbation theory to the simultaneous eigenstates of the unperturbed and perturbing Hamiltonians.

Adopting the notation introduced in Sect. 11.3, let $\psi^{(2)}_{l,s;j,m_j}$ be a simultaneous eigenstate

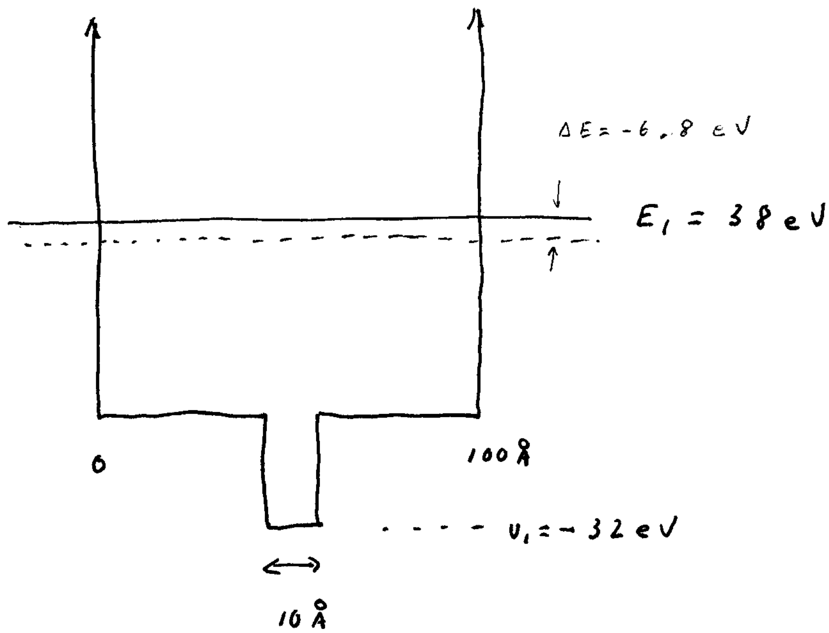


$\vec{J} = \vec{L} + \vec{S}$

$\vec{F} = \vec{J} + \vec{I}$
$= \vec{L} + \vec{S} + \vec{I}$


Hauptenergieniveau
(Bohr)
Feinstruktur
(Dirac, Lamb)
Hyperfeinstrukturaufspaltung
$n = 2$
$n = 1$
$I = \frac{1}{2}$
$(F = I \pm J)$
$P_{\frac{3}{2}} (J = \frac{3}{2})$
$S_{\frac{1}{2}} (J = \frac{1}{2})$
$P_{\frac{1}{2}} (J = \frac{1}{2})$
$S_{\frac{1}{2}} (J = \frac{1}{2})$
$F = 2$
$3\frac{A}{4}$
$f_0 = 24$ MHz
$5\frac{A}{4}$
$F = 1$
$F = 1$
$\frac{A}{4}$
$f_0 = 177$ MHz
$3\frac{A}{4}$
$F = 0$
$F = 1$
$\frac{A}{4}$
$f_0 = 59$ MHz
$3\frac{A}{4}$
$F = 0$
$F = 1$
$\frac{A}{4}$
$f_0 = 1420$ MHz
$3\frac{A}{4}$
$F = 0$

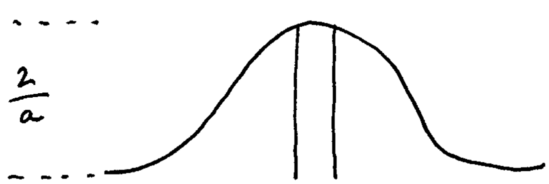
ELECTRON IN A QUANTUM WELL



$$\Delta E_m = \langle m | V_1 | m \rangle$$

$$\Delta E_m = \int V_1 \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right) dx$$

GROUND STATE SHIFT:



$$\Delta E_1 \approx \left(\frac{2}{100\,\text{Å}}\right)\left(10\,\text{Å}\right)\left(-32\,\text{eV}\right) = -6.8\,\text{eV}$$

tion will differ only slightly from the zero-order wave function, and the true eigenvalue will differ only slightly from the zero-order eigenvalue.

Figure 7.3b shows the correct shape for the true eigenfunction. The shape can be derived qualitatively by simple arguments. Near $x = L/2$, and without

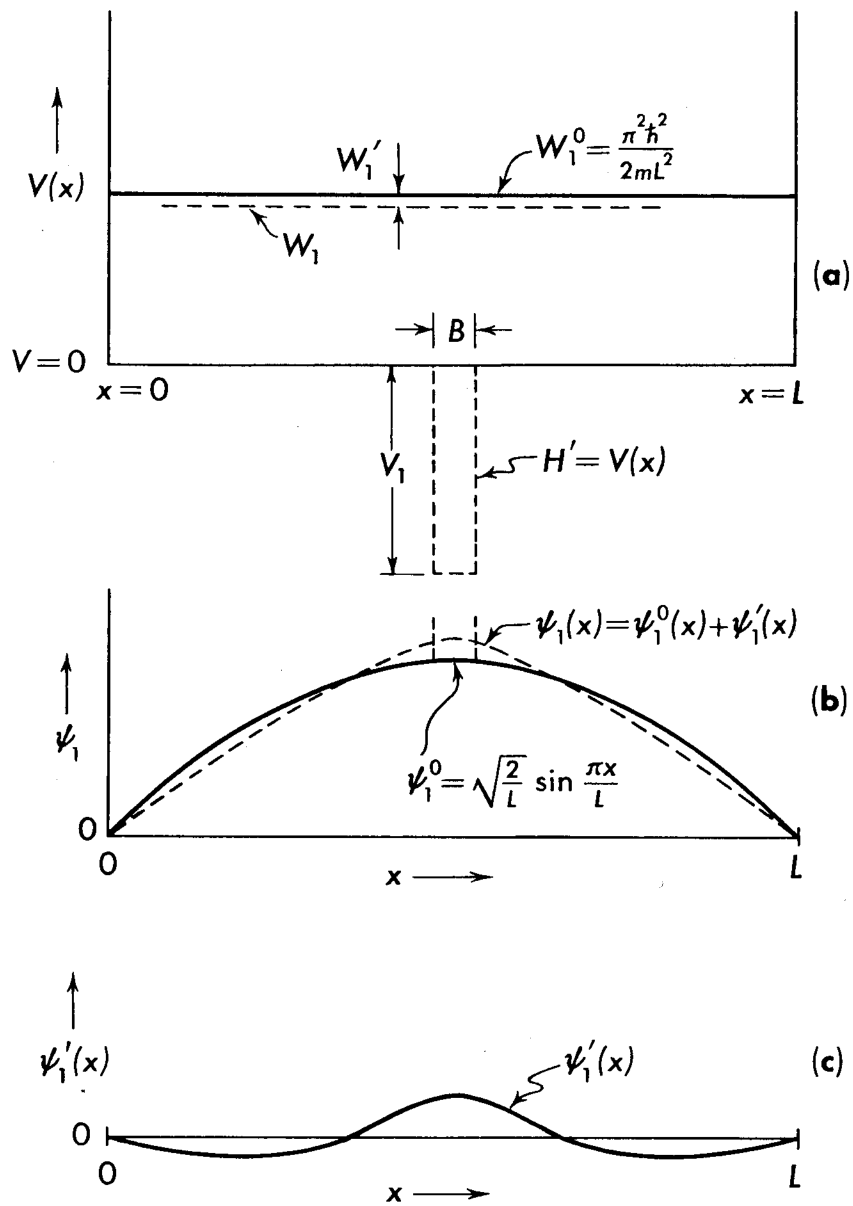


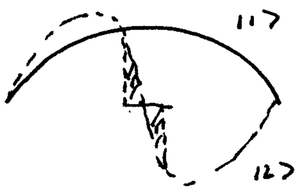
Fig. 7.3. A one-dimensional system containing a small, central potential well.

the perturbing well, the curvature of ψ, $(d^2\psi/dx^2)$ is nearly constant. When the new well is added, the curvature of ψ in the region B must be considerably greater than it was before, and therefore greater than the curvature just outside the well. This occurs since, in the region B, the difference between the potential energy and the total energy is much greater. Inside the region B the true wave

WAVE FUNCTION CHANGES

$$|m'\rangle = \sum_{\ell}{}' \frac{\langle \ell | H_1 | m \rangle}{E_m - E_\ell} |\ell\rangle$$

prime $\Rightarrow \ell \neq m$



$\Rightarrow 0$

only the ODD $|m\rangle$ CONTRIBUTE

$$|m'\rangle = \sum_{\ell} a_\ell |\ell\rangle$$

$$a_3 = \frac{\langle 3 | H_1 | 1 \rangle}{E_3 - E_1} = \frac{\int \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right) (-V_1) \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} dx}{(3^2 - 1^2)(38 eV)}$$

$$\approx \frac{\frac{2}{L}(1)(-32 \text{ eV})\, 10 Å}{(9-1)(38 eV)} \approx -0.024$$

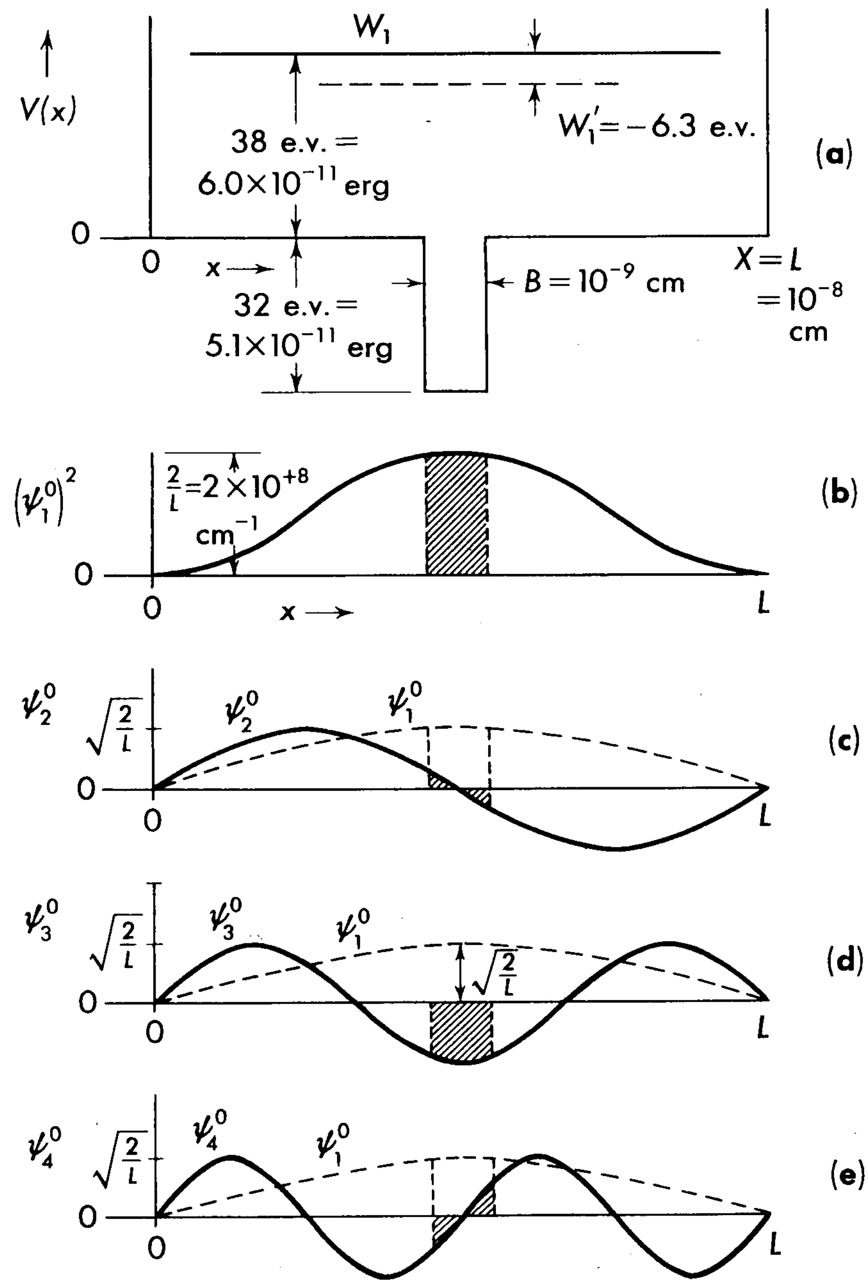


Fig. 7.4. A sample calculation using perturbation theory.

and the energy eigenvalues are,

$$W_n^0 = n^2\pi^2\hbar^2/2mL^2$$

Let the mass $= 9.11 \times 10^{-28}$ gm, $L = 10^{-8}$ cm. Since $\hbar = 1.054 \times 10^{-27}$ erg sec, we have

$$\psi_n^0 = \sqrt{2 \times 10^8} \sin n\pi x/10^{-8} \text{ (cm)}^{-(1/2)}$$

The lowest energy level is[5]

$$W_1^0 = 6\cdot 0 \times 10^{-11} \text{ erg, or } 38 \text{ e.v.}$$

[5] If $\hbar = 1.054 \times 10^{-34}$ joule sec, $m = 9.11 \times 10^{-31}$ kg, and $L = 10^{-10}$ m, then $W_1^0 = 6.0 \times 10^{-18}$ joule (1 e.v. $= 1.6 \times 10^{-19}$ joule).

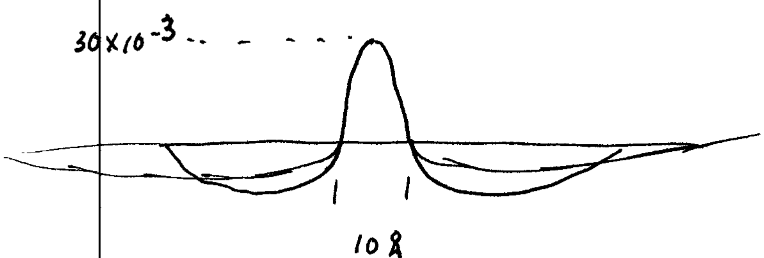
DO THE INTEGRALS

$a_3 = -20.8 \times 10^{-3}$

$a_5 = +6.2 \times 10^{-3}$

$a_7 = -2.7 \times 10^{-3}$

$a_9 = +1.3 \times 10^{-3}$



With the aid of Figure 7.4d, one can see at once that $H'_{41} = 0$, and therefore $a_4=0$.

As higher a_j's are calculated, one should use exact integration in the calculation of the intensity of the odd-numbered components, because the eigenfunctions vary more rapidly inside the perturbing well, although by symmetery

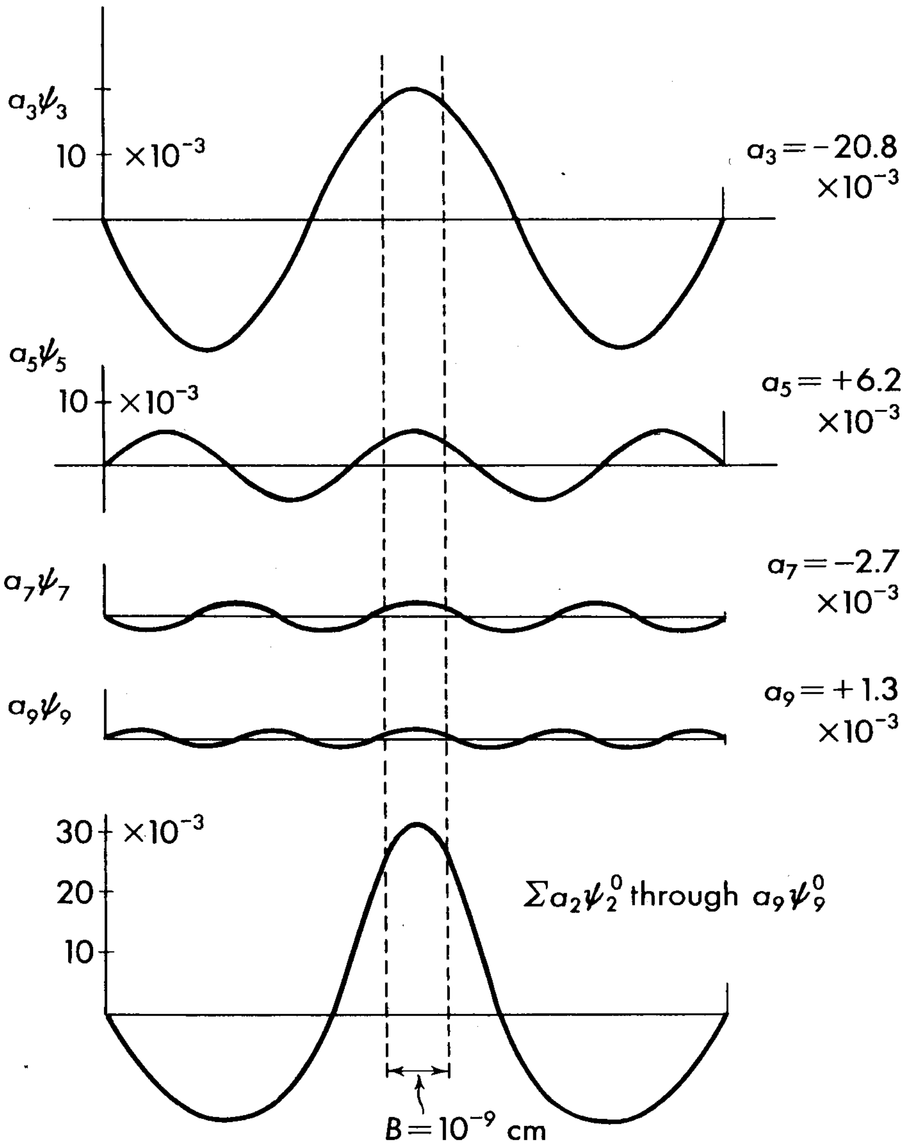


Fig. 7.5. The calculated corrections to the zero-order state ψ_1^0 of the system of Figure 7·4.

all of the even-numbered components are always exactly zero. Because the denominator $W_j^0 - W_1^0$ appears in the calculation of a_j, the magnitude of a_j becomes smaller with increasing $W_j^0 - W_1^0$.

Continuing the calculation of the a_j's, we find the amplitude of the terms up through $n = 9$. These are shown in Figure 7.5. The component wave functions are drawn to scale, with the correct sign. At the bottom of Figure

HELIUM

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}}$$

$$H_0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2}$$

$$H_1 = + \frac{e^2}{r_{12}}$$

$$\Delta E = \langle n \mid \frac{e^2}{r_{12}} \mid n \rangle$$

$$= \int \psi_{100}^* \frac{e^2}{r_{12}} \psi_{100} \, d^3r$$

$$= +33.82 \text{ eV}$$

ground state energy

$$(-Z^2 - Z^2) \, Ry = -108.24$$

FOPT

$$E_1 = -108.24 + 33.82 = -74.42$$

EXPERIMENT $E_1 = -78.62$ $\sim 5.$ % error.

From: Larry Sorensen <seattle@u.washington.edu>
Subject: How is a Fermi problem like a joke?

Hi Anon,

Your problem is a Fermi problem, but as you point out it is probably a little too easy. It is also very close to the piano tuner problem.

Finally, it doesn't have the surprise factor:

"Upon first hearing them, we don’t have even the remotest notion what the answer might be, and we feel certain that too little information has been provided to find a solution."

In jokes, the surprise factor---the punch line--- comes at the end.
In Fermi problems, the surprise factor comes at the beginning.

Try to think up something that has a big surprise factor. Of course, as you do more and more Fermi problems, you will become better and better at them, and it will be harder and harder to find big surprise factors.

You might also find it helpful to google "fermi problems" to get some more ideas:

http://en.wikipedia.org/wiki/Fermi_problem
http://www.physics.umd.edu/perg/fermi/fermi.htm
http://iws.ccccd.edu/mbrooks/demos/fermi_questions.htm
http://mathforum.org/workshops/sum96/interdisc/sheila1.html
http://en.wikipedia.org/wiki/Fermi_paradox

The creative challenge is how to pose something that has a big surprise factor.

Number of beans = (.80 x 1000 cubic centimeters)/(27/8 cubic centimeters) = approx 240 jelly beans

Solution 2

Construct or visualize a paper cube that measures 1 cubic inch.

How many jelly beans will fit in the cube?
Approximately 4

How many cubic inches are there in 1 liter?
1 inch = approx 2.54 centimeters. Therefore 1 cubic inch = approx. 16 cubic centimeters
1000 cubic centimeters/16 cubic centimeters = approx 62 cubic inches in one liter.

How many jelly beans are there in the one liter container?

62 x 4 = approximately 248 jellybeans

Fermi Questions - General Collection

1. The mass of how many Ford Mustangs is equal to the mass of the water in the Atlantic Ocean?
2. How many jelly beans fill a one-liter jar?
3. What is the mass in kilograms of the student body in your school?
4. How many golf balls will fit in a suitcase?
5. How many gallons of gasoline are used by cars each year in the United States?
6. How high would the stack reach if you piled on trillion dollar bills in a single stack?
7. Approximately what fraction of the area of the continental United States is covered by automobiles?
8. How many hairs are on your head?
9. What is the weight of solid garbage thrown away by American families every year?
10. If your life earnings were doled out to you at a certain rate per hour for every hour of your life, how much is your time worth?

11. How many cells are there in the human body?

12. How many individual frames of film are needed for a feature-length film? How long is such a film?

13. How many water balloons will it take to fill the school gymnasium?

14. How many flat toothpicks would fit on the surface of a sheet of poster board?

15. How many hot dogs will be eaten at major league baseball games during a one year season?

16. How many revolutions will a wheel on the bus make during our seventh grade trip from Baton Rouge, LA to Washington, D.C.?

17. How many minutes will be spent on the phone by middle school students in the United States?

18. How many pizzas will be ordered in your state this year?

1.

The Fermi Solution

At twenty-nine minutes past five, on a Monday morning in July of 1945, the world's first atom bomb exploded in the desert sixty miles northwest of Alamogordo, New Mexico. Forty seconds later, the blast's shock wave reached the base camp, where a cluster of scientists stood in stunned contemplation of the historic spectacle. The first person to stir was the Italian-American physicist Enrico Fermi, who was on hand to witness the culmination of a project he had helped to initiate.

Before the bomb detonated, Fermi had torn a sheet of notebook paper into small bits. Then, as he felt the first quiver of the shock wave spreading outward through the still air, he released the shreds above his head. They fluttered down and away from the mushroom cloud growing on the horizon, landing about two and a half yards behind him. After a brief mental calculation, Fermi announced that the bomb's energy had been equivalent to that produced by ten thousand tons of TNT. Sophisticated instruments were also at the site, and analysis of their readings of the shock wave's velocity and pressure, an exercise that took several weeks to complete, confirmed Fermi's instant estimate.

The bomb-test team was impressed, but not surprised, by this brilliant bit of scientific improvisation. Enrico Fermi's genius was known throughout the world of physics. In 1938 he had won a Nobel Prize for his work on elementary particles, and four years later, in Chicago, had produced the first sustained nuclear chain reaction, thereby ushering in the age of atomic weapons and commercial nuclear power. No other physicist of his generation, and no one since, has been at once a masterly experimentalist and a leading theoretician. In miniature, the bits of paper and the analysis of their motion exemplified this unique combination of gifts.

Like all virtuosos, Fermi had a distinctive style. His approach to physics brooked no opposition; it simply never occurred to him that he might fail to find the solution to a problem. His scientific papers and books reveal a disdain for embellishment—a preference for the most direct, rather than the most intellectually elegant, route to an answer. When he reached the limits of his cleverness, Fermi completed a task by brute force.

To illustrate this approach, imagine that a physicist must determine the volume of an irregular object—say, Earth, which is slightly pear-shaped. He might feel stymied without some kind of formula, and there are several ways he could go about getting one. He could consult a mathematician, but finding one with enough knowledge and interest to be of help might be difficult. He could search through the mathematical literature, a time-consuming and probably fruitless exercise because the ideal shapes that interest mathematicians often do not match those of the irregular objects found in nature. Or he could set aside his own research in order to derive the formula from basic mathematical principles, but, of course, if he had wanted to devote his time to theoretical geometry, he wouldn't have become a physicist.

Alternatively, the physicist could do what Fermi would have done—compute the volume numerically. Instead of relying on a formula, he could mentally divide the planet into a large number of tiny cubes, each with a volume easily determined by multiplying the length times the width times the height, and then add together the answers to these more tractable problems. This method yields only an approximate solution, but it is sure to produce the desired result, which is what mattered to Fermi. With the introduction of computers after the Second World War and, later, of pocket calculators, numerical computation has become standard procedure in physics.

The technique of dividing difficult problems into small, manageable ones applies to many problems besides those amenable to numerical computation. Fermi excelled at this rough-and-ready modus operandi, and, to pass it on to his students, he developed a type of question that has become associated with his name. A Fermi problem has a characteristic profile: Upon first hearing it, one doesn't have even the remotest notion of what the answer might be, and one feels certain that too little information has been provided to find a solution. Yet, when the problem is broken down into subproblems, each one answerable without the help of experts or reference books, an estimate can be made, either mentally or on the back of an envelope, that comes remarkably close to the exact solution.

Suppose, for example, that one wants to determine Earth's circumference without looking it up. Everyone knows that New York and Los Angeles are separated by about three thousand miles and that the time difference between the two coasts is three hours. Three hours corresponds to one eighth of a day, and a day is the time it takes the planet to complete one revolution, so its circumference must be eight times three thousand, or twenty-four thousand miles—an answer that differs from the true value (at the equator, 24,902.45 miles) by less than 4 percent. In John Milton's words:

> so easy it seemed
> Once found, which yet unfound most
> would have thought
> Impossible.

Fermi problems might seem to resemble the brainteasers that appear among the back pages of airline magazines and other popular publications (Given three containers that hold eight, five, and three quarts, respectively, how do you measure out a single quart?), but the two genres differ significantly. The answer to a Fermi problem, in contrast to that of a brainteaser, cannot be verified by logical deduction alone and is always approximate. (To determine earth's circumference precisely, the planet must actually be measured.) Then, too, solving a Fermi problem requires a knowledge of facts not mentioned in the statement of the problem. (In contrast, the decanting puzzle contains all the information necessary for its solution.)

These differences mean that Fermi problems are more closely tied to the physical world than mathematical puzzles, which rarely have anything practical to offer physicists. By the same token, Fermi problems are reminiscent of the ordinary dilemmas that nonphysicists encounter every day of their lives. Indeed, Fermi problems and the way they are solved not only are essential to the practice of physics, but also teach a valuable lesson in the art of living.

How many piano tuners are there in Chicago? The whimsical nature of this question, the improbability that anyone knows the answer, and the fact that Fermi posed it to his classes at the University of Chicago have elevated it to the status of legend. There is no standard solution (that's exactly the point), but anyone can make assumptions that quickly lead to an approximate answer. Here is one way: If the population of metropolitan Chicago is three million, an average family consists of four people, and one third of all families own pianos, there are two hundred and fifty thousand pianos in the city. If every piano is tuned once every five years, fifty thousand pianos must be tuned each year. If a tuner can service four pianos a day, two hundred and fifty days a year, for a total of one thousand tunings a year, there must be about fifty piano tuners in the city. The answer is not exact; it could be as low as twenty-five or as high as a hundred. But, as the yellow pages of the telephone directory attest, it is definitely in the ballpark.

Fermi's intent was to show that although, at the outset, even the answer's order of magnitude is unknown, one can proceed on the basis of different assumptions and still arrive at estimates that fall within range of the answer. The reason is that, in any string of calculations, errors tend to cancel one another out. If someone assumes, for instance, that every sixth, rather than third, family owns a piano, they are just as likely to assume that pianos are tuned twice in five years, instead of once. It is as improbable that all of one's errors will be underestimates (or overestimates) as it is that all the throws in a series of coin tosses will be heads (or tails). The law of probabilities dictates that deviations from the correct assumptions will tend to compensate for one another, so the final results will converge toward the right number.

Of course, the Fermi problems that physicists face deal more often with atoms and molecules than with pianos. To answer them, one needs to commit to memory a few basic magnitudes, such as the approximate radius of a typical atom or the number of molecules in a thimbleful of water. Equipped with such facts, one can estimate, for example, the distance a car must travel before a layer of rubber about the thickness of a molecule is worn off the tread of its tires. It turns out that that much is removed with each revolution of the wheels, a reminder of the immensity of the number of atoms in a tire. (Assume that the tread is about a quarter-inch thick and that it wears off in forty thousand miles of driving. If a quarter inch is divided by the number of revolutions a typical wheel, six feet in circumference, makes in forty thousand miles, the answer is roughly a hundred millionth of an inch, or a molecular diameter.)

More momentous Fermi problems might concern energy policy (the number of solar cells required to produce a certain amount of electricity), environmental quality (the amount of acid rain caused annually by coal consumption in the United States), or military technology. A good example from the weapons field was proposed in 1981 by David Hafemeister, a physicist at the California Polytechnic State University: For what length of time would the beam from the most powerful laser have to be focused on the skin of an incoming missile to ignite the chemical explosives in the missile's nuclear warhead? The key point is that a beam of light, no matter how well focused, spreads out like an ocean wave entering the narrow opening of a harbor, a phenomenon called diffraction broadening. The formula that describes such spreading applies to all forms of waves, including light waves, so, at a typical satellite-to-missile distance of, perhaps, seven hundred miles, a laser's energy will become considerably attenuated. With some reasonable assumptions about the temperature at which explosive materials ignite (say, a thousand degrees Fahrenheit), the diameter of the mirror that focuses the laser beam (ten feet is about right), and the maximum available power of chemical lasers (a level of a million watts is conceivable), the answer turns out to be around ten minutes.

Trying to keep a laser aimed at a speeding missile at a distance of seven hundred miles for ten minutes is a task that greatly exceeds the capacity of existing technology. For one thing, the missile travels so rapidly that it would be impossible to keep it within the laser's range. For another, a laser beam must reflect back toward its source to verify that it is hitting its target, which would be comparable to aiming a flashlight at a small mirror carried by a running man at the opposite end of a football field in such a way that the light reflected from the mirror would shine back into one's eyes.

The solution of this Fermi problem depends on more facts than average people, or even average physicists, have at their fingertips, but for those who do have them in mind, the calculation takes only a few minutes, and produces a result that is no less accurate for being easy to perform. Therefore, Hafemeister's simple conclusion, which predated President Reagan's 1983 Star Wars speech, agrees roughly with the findings of the American Physical Society's 1987 report entitled *Science and Technology of Directed Energy Weapons,* which was the result of much more elaborate analysis. Prudent physicists—those who want to avoid false leads and dead ends—operate according to a long-standing principle: Never start a lengthy calculation until you know the range of values within which the answer is likely to fall (and, equally important, the range within which the answer is *un*likely to fall). They attack every problem as a Fermi problem, estimating the order of magnitude of the result before engaging in an investigation.

Physicists also use Fermi problems to communicate with one another. When they gather in university hallways, convention-center lobbies, or cozy restaurants to discuss a new experiment or theory, they often first survey the territory by staking out, in a numerical way, the perimeter of the problem at hand. Only the timid hang back, deferring to the experts in their midst. Those accustomed to tackling Fermi problems approach the subject as if it were their own, demonstrating their understanding by performing rough calculations. If the conversation turns to a new particle accelerator, for example, they will estimate the strength of the magnets it requires; if the subject is the structure of a novel crystal, they will calculate the spacing between its atoms. Everyone tries to arrive at the correct answer with the least amount of effort. It is this spirit of independence, which he himself possessed in ample measure, that Fermi sought to instill by posing his unconventional problems.

Questions about atom bombs, piano tuners, automobile tires, laser weapons, particle accelerators, and crystal structure have little in common. But the means by which they are answered is the same in every case, and can be applied with equal success to questions outside the realm of physics. Whether the problem concerns cooking, automobile repair, or personal relationships, there are two basic types of responses: the fainthearted turn to authority—to reference books, bosses, expert consultants, physicians, ministers—while the independent of mind delve into that private store of common sense and limited factual knowledge that everyone carries, make reasonable assumptions, and derive their own, admittedly approximate, solutions. Stripped transmissions and severe depressions usually require professional help but more mundane challenges—preparing chili from scratch, replacing a water pump, resolving a family quarrel—can often be sorted out with nothing more than logic, common sense, and patience.

The similarities between technical problems and human ones is explored in Robert M. Pirsig's 1974 book, *Zen and the Art of Motorcycle Maintenance,* in which the repair and upkeep of a machine serves as a metaphor for rationality itself. At one point the protagonist proposes to fix the slipping handlebars of a friend's new BMW motorcycle, the pride of a half century of German mechanical craftsmanship, with a piece of an old beer can. Although the proposal happens to be technically perfect (the aluminum is thin and flexible), the cycle's owner, a musician, cannot break his reliance on authority; since the idea did not come from a factory-trained mechanic, it does not deserve serious consideration. In the same way, certain observers would have been skeptical of Fermi's analysis, carried through with the aid of a handful of confetti, of a two-billion-dollar bomb test. Such an attitude demonstrates less, perhaps, about their knowledge of the problem than about their attitude toward life. As Pirsig put it, "The real cycle you're working on is a cycle called 'yourself.' "

Ultimately, the value of dealing with the problems of science, or those of everyday life, in the way Fermi did lies in the rewards one gains for making independent discoveries and inventions. It doesn't matter whether the discovery is as momentous as the determination of the yield of an atom bomb or as insignificant as an estimate of the number of piano tuners in a Midwestern city. Looking up the answer, or letting someone else find it, actually impoverishes one; it robs one of the pleasure and pride that accompany creativity and deprives one of an experience that, more than anything else in life, bolsters self-confidence. Self-confidence, in turn, is the essential prerequisite for solving Fermi problems. Thus, approaching personal dilemmas as Fermi problems can become, by a kind of chain reaction, a habit that enriches life.